Suman Patra, Assistant Professor, Department of Physics,

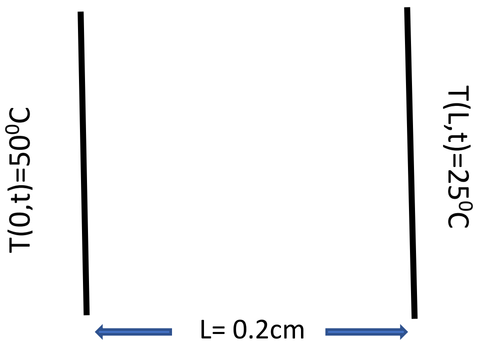
Netaji Nagar Day College

Topic for

Semester – 4, Paper – PHSA CC8

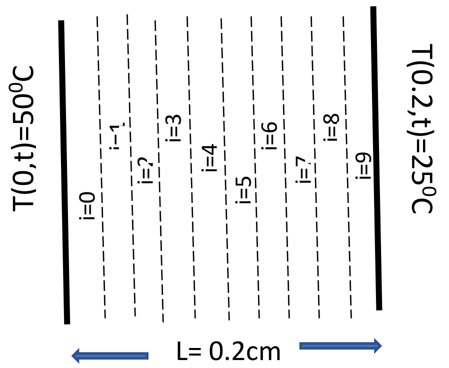
SOLVING 1D HEAT DIFFUSION EQUATION NUMERICALLY

Let's say we have a plane wall with a certain surface temperature $T=50^0C$ on one side and have a different surface temperature $T=25^0C$ on another side. Let's assume thickness of the wall is L= 0.2 cm.



We will now break this plane wall into discrete sections of thickness Δx=0.02cm.

So, we are basically dividing the wall by n=10 discrete elements.



Now denote the leftmost element at x=0 by i=0, the next element by i+1=1 and so on. The rightmost element is denoted by i=n-1=9.

Let us now write the heat diffusion equation.

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

Now we will discretize this equation.

We can write, $\frac{\partial T}{\partial t} = \frac{T(t+\Delta t)-T(t)}{\Delta t}$ Say Δt= dt =0.1

Now, $\frac{\partial^2 T}{\partial x^2} = \frac{\frac{\partial T}{\partial x}(at\ x) - \frac{\partial T}{\partial x}(at\ x-\Delta x)}{\Delta x}$

$$= \frac{\frac{T(x+\Delta x)-T(x)}{\Delta x} - \frac{T(x)-T(x-\Delta x)}{\Delta x}}{\Delta x} = \frac{T(x+\Delta x) - 2T(x) + T(x-\Delta x)}{(\Delta x)^2}$$

Hence, we can write

$$\frac{\boldsymbol{T}(\boldsymbol{t}+\Delta \mathbf{t}) - \boldsymbol{T}(\boldsymbol{t})}{\Delta \mathbf{t}} = \boldsymbol{\alpha} \frac{\boldsymbol{T}(\boldsymbol{x}+\Delta \mathbf{x}) - \mathbf{2}\boldsymbol{T}(\boldsymbol{x}) + \boldsymbol{T}(\boldsymbol{x}-\Delta \mathbf{x})}{(\Delta \mathbf{x})^2}$$

The heat equation is discretized now.

Let’s rearrange this equation.

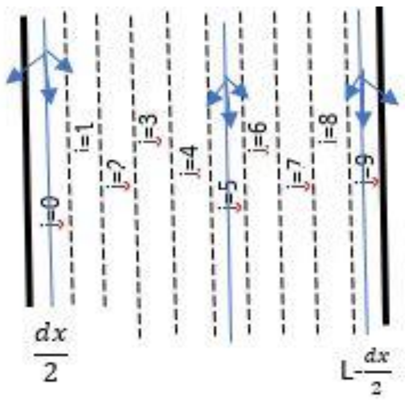
$$T(t+\Delta t) - T(t) = \frac{\alpha\, \Delta t}{(\Delta x)^2} * \{T(x+\Delta x) - 2T(x) + T(x-\Delta x)\}$$

Therefore,

$$T(t+\Delta t) = T(t) + \frac{\alpha\, \Delta t}{(\Delta x)^2} * \{T(x+\Delta x) - 2T(x) + T(x-\Delta x)\}$$

Let’s say $\frac{\alpha}{(\Delta x)^2} * \{T(x+\Delta x) - 2T(x) + T(x-\Delta x)\} = \delta T$

$\therefore T(t+\Delta t) = T(t) + \delta T * \Delta t$



Now let's try to calculate the temperature T of the i-th (located at x) element at any instant of time $t + \Delta t$.

$$\boldsymbol{T(i, t+\Delta t) = T(i,t) + \frac{\alpha\,\Delta t}{(\Delta x)^2} * \{T(i+1,t) - 2T(i,t) + T(i-1,t)\}}$$

We can see that to calculate the temputure T of i-th element at any instant $t + \Delta t$ we need to know the temputure of i-th element, (i+1)th element and (i-1)th element at a previous instant t.

For eaxmple, if we want to calculate the temputure T of 5th element at any instant $t + \Delta t$ we need to know the temputure of 5th element, 6th element and 4th element at a previous instant t.

We have assumed that Δt= dt =0.1

So, if we know the temperature of all the elements at t=0, then we can calculate the temperature of all elements at t= 0+0.1 = 0.1

Using the temperature data at t=0.1 we can calculate the temperature of all elements at t= 0.1+0.1 = 0.2

So, step by step we can calculate temperature of all the elements at any instant of time t.

Again,

$$T(i, t+\Delta t) = T(i,t) + \delta T(i,t) * \Delta t$$

$$T(i, t+\Delta t) = T(i,t) + \frac{\alpha\,\Delta t}{(\Delta x)^2} * \{T(i+1,t) - 2T(i,t) + T(i-1,t)\}$$

$$\therefore \boldsymbol{\delta T(i,t) = \frac{\alpha}{(\Delta x)^2} * \{T(i+1,t) - 2T(i,t) + T(i-1,t)\}}$$

When i=0; i+1=1, i-1=-1

i=1; i+1=2, i-1=0

…………………………

i=8; i+1=9, i-1=7

i=9; i+1=10, i-1=8

Hence, for i=1 to i=n-2=8 δT(i,t) can be calculated inside the loop incrementing [i].

For i=0 and i=n-1=9 δT(i,t) { δT(0,t) & δT(9,t)} should be calculated separately.

δT(0,t)= $\frac{\alpha}{(\Delta x)^2} * \{T(1,t) - 2T(0,t) + T(-1,t)\}$

δT(9,t)= $\frac{\alpha}{(\Delta x)^2} * \{T(10,t) - 2T(9,t) + T(8,t)\}$

But there is no such element denoted by i=-1 or i=10!

But we can easily understand that the element i=-1 (if it exists) must be located at left of the element i=0. So, the element i=-1 must be located inside the left boundary of the wall. Left boundary of the wall has surface temperature T(0,t)=50^{0}C.

$\therefore T(-1,t) =$ T(0,t)= 50^{0}C

Similarly, we can easily understand that the element i=10 (if it exists) must be located at right of the element i=9. So, the element i=10 must be located inside the right boundary of the wall. Right boundary of the wall has surface temperature T(L,t)=25^{0}C.

$\therefore T(10,t) =$ T(L,t)= 25^{0}C

We now know how to approach 1D heat equation numerically.

We should choose the number of elements (n) depending upon the length (L) of the wall. (**Smaller value of Δx means higher accuracy)

We should also choose the value of Δt as small as possible. (**Smaller value of Δt means higher accuracy)

T(i,t=0)=T_0 {for all values of i} will be provided as initial condition.

Using this data T(i,t=0.1) {for all values of i} can be calculated easily.

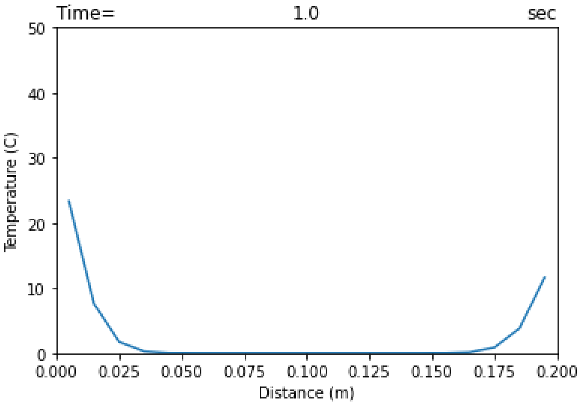
So, we will need a loop incrementing [t] up to the desired value of t (Let's say 10 sec). Inside this loop incrementing [t] we will also need a loop incrementing [i] which in turn will calculate the values of T(i,t) for all the elements.

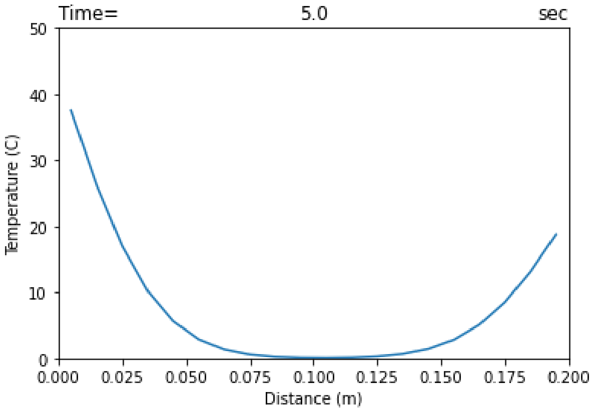
Without wasting any more time let's select any of our preferred language (C, Fortran, Matlab, Python) and develop the source code.

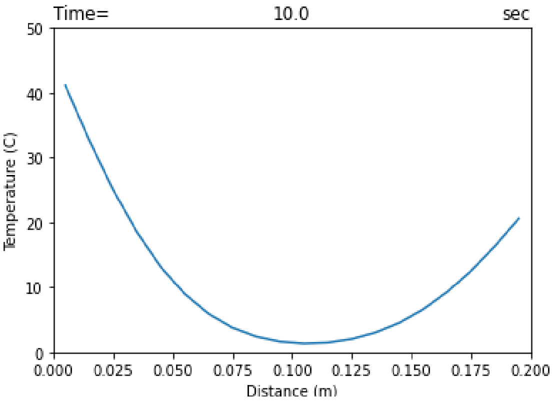
Python Code

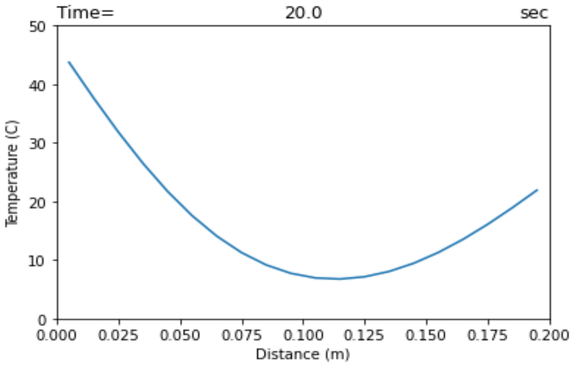
```
#1D Heat Diffusion Equation
import numpy as np
import matplotlib.pyplot as plt

L=0.2 #Thickness of wall/rod
n=20 #no of elements in wall/rod
T0=0 #Initial temperature
TRL=50 #Surface temperature at left side of the wall
TRR=25 #Surface temperature at right side of the wall
alpha=0.0001 #Thermal diffusivity, material specific
dx=L/n #Element size
t_final=60 #Final time at which you want to see the temperature distribution
dt=0.1 #Time interval

x=np.linspace(dx/2,L-dx/2,n) #Constructing position matrix

T=np.ones(n)*T0 #Constructing Temperature matrix at any instant of time

dT=np.empty(n) #Constructing matrix to compute increase in temperature along x-axis

t=np.arange(0,t_final,dt) #Constructing time matrix

for j in range(1,len(t)):
        plt.clf()
        for i in range(1,n-1):
                dT[i]=alpha*((T[i+1]-2*T[i]+T[i-1])/dx**2)
        dT[0]=alpha*((T[1]-2*T[0]+TRL)/dx**2)
        dT[n-1]=alpha*((TRR-2*T[n-1]+T[n-2])/dx**2)
        T=T+dT*dt
        plt.figure(1)
        plt.title("Time=",loc='left')
        plt.title((j+1)*dt,loc='center')
        plt.title("sec",loc='right')
        plt.plot(x,T)
        plt.axis([0,L,0,50])
        plt.xlabel('Distance (m)')
        plt.ylabel('Temperature (C)')

        plt.show()
        plt.pause(0.01)

print('Thanks')
```

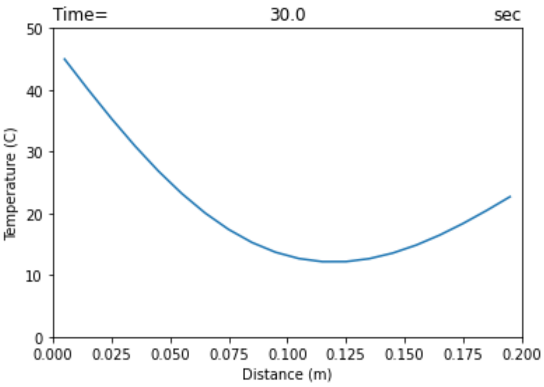
<u>Output</u>

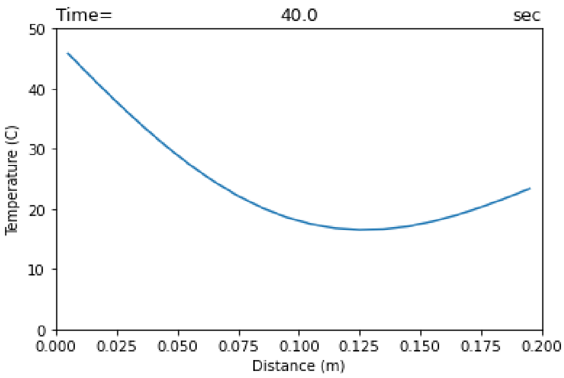

Time= 1.0 sec
Temperature (C)
Distance (m)

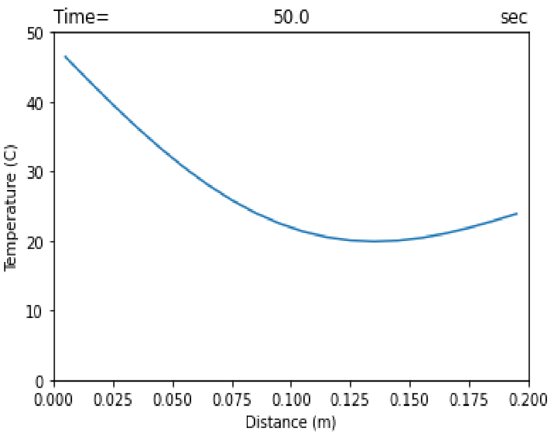

Time= 5.0 sec
Temperature (C)
Distance (m)

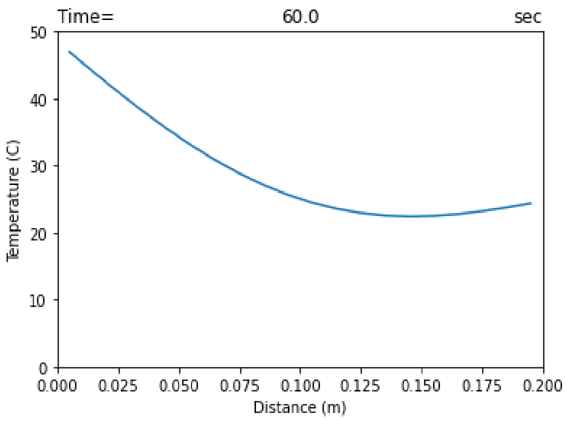

Time= 10.0 sec
Temperature (C)
Distance (m)


Time= 20.0 sec
Temperature (C)
Distance (m)


Time= 30.0 sec
Temperature (C)
Distance (m)


Time= 40.0 sec
Temperature (C)
Distance (m)


Time= 50.0 sec
Temperature (C)
Distance (m)


Time= 60.0 sec
Temperature (C)
Distance (m)